Jour. Ind. Soc. Ag. Statistics
Vol. XXXXI, No. 1 (April, 1989), pp.69-76

CONCISE ESTIMATORS OF BIAS AND VARIANCE OF THE FINITE POPULATION CORRELATION COEFFICIENT

R. S. RANA
Punjab Agricultural University, Ludhiana
(Received : February, 1984)

SUMMARY

The paper presents short and appealing expressions for bias, variance and estimate of variance of r, the sample correlation coefficient in finite population.

Keywords : Finite population, Correlation coefficient, Bivariate normal distribution.

Introduction

The study of correlation coefficient is frequent in the fields of Psychology, Sociology and Economics where populations are finite. So an estimator of finite population correlation coefficient is desirable. Gupta, Singh and Lal [2] attempted this problem. In this article, the author reports short and appealing expressions for bias, variance and estimate of variance of r.

2. The Estimator

Let $\{(x_i, y_i);\ i = 1, 2, \ldots, n\}$ be pairs of observations on n units of a sample from a finite population of N units. Notations to be followed in this paper are

$$\Sigma \text{ for } \sum_{i=1}^{N};\ \Sigma' \text{ for } \sum_{i=1}^{n};\ Z_i = X_i Y_i;\ \bar{X} = \Sigma X_i/N;\ \bar{x} = \Sigma' x_i/n;$$

$$S_{x^2}^2 = (\Sigma(X_i^2)^2 - (\Sigma X_i^2)^2/N)/(N-1);$$

$$S_{xy} = (\Sigma X_i Y_i^2 - (\Sigma X_i)(\Sigma Y_i^2)/N)/(N-1);$$

$$\sigma^2_{x^2} = (N-1)(S^2_{x^2}/N);\ \sigma_{xy^2} = (N-1)\, S_{xy^2}/N;$$

$$s^2_{x^2} = (\Sigma' (x_i^2)^2 - (\Sigma' x_i^2)^2/n)/(n-1) \text{ and}$$

$$s_{xy^2} = (\Sigma' x_i y_i^2 - (\Sigma' x_i)(\Sigma' y_i^2)/n)/(n-1)$$

Similarly expressions for $\overline{Y}$, $\bar{y}$, S_{zy}, σ_{zy}, s_{zy}, $S^2_{zy^2}$, σ_{zy^2}, s^2_{zy}, S_{xz}, σ_{xz}, s_{xz}, S_{zx^2}, σ_{zx^2}, s_{zx^2} etc. are followed.

The usual estimator r of ρ, the population correlation coefficient is

$$r = \hat{\theta}_1/(\hat{\theta}_2\,\hat{\theta}_3)^{1/2} \tag{1}$$

where

$$\hat{\theta}_1 = \Sigma' x_i y_i - (\Sigma' x_i)(\Sigma' y_i)/n,$$

$$\hat{\theta}_2 = \Sigma' x_i^2 - (\Sigma' x_i)^2/n \quad \text{and}$$

$$\hat{\theta}_3 = \Sigma' y_i^2 - (\Sigma' y_i)^2/n \tag{2}$$

Let $\hat{\theta}_i = \theta_i + \varepsilon_i$ $(i = 1, 2, 3)$ such that $E(\hat{\theta}_i) = \theta_i$. So in case of simple random sampling without replacement (SRSWOR)
$\theta_1 = (n-1)\, S_{xy}$, $\theta_2 = (n-1)\, S_x^2$ and $\theta_3 = (n-1)\, S_y^2$.

LEMMA 1 : For SRSWOR with $n > 1$ and $N > 3$, the variance of ε_1 will be

$$V(\varepsilon_1) = \alpha\,(S_z^2 - 2\overline{X}\,S_{zy} - 2\,\overline{Y}S_{zx} + \overline{X}^2\,S_y^2 + \overline{Y}^2\,S_x^2$$

$$+ 2\,\overline{X}\overline{Y}S_{xy}) + \beta\left(\frac{N-1}{N-3}S_x^2\,S_y^2 + S_{xy}^2\right) \tag{3}$$

where $\alpha = \dfrac{(N-n)(n-1)(nN-N-n-1)}{n(N-2)(N-3)}$

and $\beta = \dfrac{(N-n)(n-1)(N-n-1)}{nN(N-2)}$

COROLLARY 1 : For SRSWOR with $n > 1$ and $N > 3$

$$V(\varepsilon_2) = \alpha\,(S^2_{x2} - 4\,\overline{X}S_{xx^2} + 4\,\overline{X}^2\,S_x^2) + 2\beta\,\frac{(N-2)}{(N-3)}\,S_x^4 \tag{4}$$

COROLLARY 2 : For SRSWOR with $n > 1$ and $N > 3$

$$V(\varepsilon_3) = \alpha\,(S^2_{y^2} - 4\,\overline{Y}S_{yy^2} + 4\,\overline{Y}^2\,S_y^2) + 2\,\frac{\beta(N-2)}{(N-3)}\,S_y^4 \tag{5}$$

LEMMA 2 : The covariance between ε_1 and ε_2 will be

$$\text{Cov}\ (\varepsilon_1, \varepsilon_2) = \alpha(S_{zx^2} - \overline{Y}_{xx^2} - 3\overline{X}S_{xz} + \overline{X}^2\,S_{xy}$$

$$+ 3\overline{X}\overline{Y}S_x^2) + \frac{2\beta\,(N-2)}{N-3}\,S_{xy}\,S_x^2 \tag{6}$$

COROLLARY 3 : The covariance between ε_1 and ε_3 will be

$$\text{Cov}\ (\varepsilon_1, \varepsilon_3) = \alpha(S_{xy^2} - \bar{X}S_{yy^2} - 3\bar{Y}S_{yx} + \bar{Y}^2 S_{xy} + 3\bar{X}\bar{Y}S_y^2) + \frac{2\beta\,(N-2)}{N-3}\, S_{xy}\, S_y^2 \tag{7}$$

LEMMA 3 : The covariance between ε_2 and ε_3 will be

$$\text{Cov}\ (\varepsilon_2, \varepsilon_3) = \alpha(S_{x^2y} - 2\,\bar{X}S_{xy^2}\ \ \bar{Y}S_{yx^2} + 4\bar{X}\bar{Y}S_{xy}) + \frac{2\beta}{N-3}\,((N-1)\ S_{xy}^2 - S_x^2\, S_y^2) \tag{8}$$

The lemmas 1-3 can be easily proved with little algebraic manipulations of the respective expressions reported by Gupta, Singh and Lal [2].

LEMMA 4 : For SRSWR, the variance of ε_1 will be

$$V(\varepsilon_1) = \frac{(n-1)^2}{n}\,(\sigma_x^2 + \bar{X}^2\sigma_y^2 + \bar{Y}^2\sigma_x^2 - 2\bar{X}\,\sigma_{yx} - 2\bar{Y}\sigma_{xx} + 2\bar{X}\bar{Y}\sigma_{xy}) + \frac{(n-1)}{n}\,(\sigma_{xy}^2 + \sigma_y^2\,\sigma_x^2) \tag{9}$$

COROLLARY 4 : For SRSWR, the variance of ε_2 will be

$$V(\varepsilon_2) = \frac{(n-1)^2}{n}\,(\sigma_{x^2}^2 - 4\bar{X}\sigma_{xx^2} + 4\bar{X}^2\sigma_x^2) + \frac{2\,(n-1)}{n}\,\sigma_x^4 \tag{10}$$

COROLLARY 5 : For SRSWR, the variance of ε_3 will be

$$V(\varepsilon_3) = \frac{(n-1)^2}{n}\,(\sigma_{y^2}^2 - 4\bar{Y}\,\sigma_{yy^2} + 4\bar{Y}^2\sigma_y^2) + \frac{2\,(n-1)}{n}\,\sigma_y^4 \tag{11}$$

LEMMA 5 : For SRSWR, the covariance between ε_1 and ε_2 will be

$$\text{Cov}\ (\varepsilon_1, \varepsilon_2) = \frac{(n-1)^2}{n}\,(\sigma_{zx^2} - 3\bar{X}\,\sigma_{xz} - \bar{Y}\,\sigma_{xx^2} + \bar{X}^2\,\sigma_{xy} + 3\bar{X}\bar{Y}\,\sigma_x^2) + \frac{2\,(n-1)}{n}\,\sigma_{xy}\,\sigma_x^2 \tag{12}$$

COROLLARY 6 : For SRSWR, the covariance between ε_1 and ε_3 will be

$$\text{Cov}\ (\varepsilon_1, \varepsilon_3) = \frac{(n-1)^2}{n}\,(\sigma_{zy^2} - 3\bar{Y}\,\sigma_{yz} - \bar{X}\sigma_{yy^2} + \bar{Y}^2\,\sigma + 3\bar{X}\bar{Y}\sigma_y^2) + \frac{2\,(n-1)}{n}\,\sigma_{xy}\,\sigma_y^2 \tag{13}$$

LEMMA 6 : For SRSWR, the covariance betweer and ε_3 will be

$$\text{Cov}\ (\varepsilon_2, \varepsilon_3) = \frac{(n-1)^2}{n}\,(\sigma_{x^2y^2} - 2\bar{X}\,\sigma_{xy^2} - 2\bar{Y}\,\sigma_{yx^2} + 4\bar{X}\bar{Y}\,\sigma_{xy}) + \frac{2\,(n-1)}{n}\,\sigma_{xy}^2 \tag{14}$$

Lemma 4 can be proved with algebraic manipulations of expressions of $V(\hat{\theta}_1)$ given by Gupta, Singh and Lal [2] (corollary 6, pp. 46). Similarly lemmas 5 and 6 are proved.

3. Bias of the Usual Estimator r

As r is biased estimator of ρ, the population correlation coefficient, it is imperative to work out its bias.

THEOREM 1 : *For SRSWOR; the bias of r is*

$$\begin{aligned} B(r) \doteq & -\rho\alpha \left\{ \frac{1}{2\theta_1\theta_2}(S_{zx^2} - 3\bar{X}S_{xz}) + \frac{1}{2\theta_1\theta_3}(S_{zy^2} - 3\bar{Y}S_{yz}) \right. \\ & + \frac{1}{4\theta_2\theta_3}(2\bar{X}S_{xy^2} + 2\bar{Y}S_{yx^2} - S^2_{xy^2}) - \frac{3}{8}\left(\frac{S^2_{x^2}}{\theta_2^2} + \frac{S^2_{y^2}}{\theta_3^2}\right) \\ & + \frac{1}{2\theta_2}(3\bar{X}/\theta_2 - \bar{Y}/\theta_1) S_{xx^2} + (\bar{X}^2/2\theta_1\theta_2 + \bar{Y}^2/2\theta_1\theta_3 \\ & - \bar{X}\bar{Y}/\theta_2\theta_3) S_{xy} + \frac{3}{2\theta_2}(\bar{Y}/\theta_1 - \bar{X}/\theta_2)\bar{X}S_x^2 + \frac{3}{2\theta_3}(\bar{X}/\theta_1 \\ & \left. - \bar{Y}/\theta_3)\bar{Y}S_y^2 + \frac{1}{2\theta_3}(3\bar{Y}/\theta_3 - \bar{X}/\theta_1) S_{yy^2} \right\} \\ & - \frac{\rho\beta(N-2)}{(N-3)} \{ (S_x^2/\theta_2 + S_y^2/\theta_3) S_{xy}/\theta_1 - \frac{1}{2\theta_2\theta_3(N-2)} \\ & ((N-1)S^2_{xy} - S_x^2 S_y^2) - \frac{3}{4}(S_x^4/\theta_2^2 + S_y^4/\theta_3^2)\} \qquad (15) \end{aligned}$$

Proof: We have $r = \hat{\theta}_1/(\hat{\theta}_2\hat{\theta}_3)^{1/2} \doteq \rho(1 + \varepsilon_1/\theta_1)(1 + \varepsilon_2/\theta_2)^{-1/2}$ $(1 + \varepsilon_3/\theta_3)^{-1/2}$.

Assuming $|\varepsilon_2/\theta_2| < 1$ and $|\varepsilon_3/\theta_3| < 1$ and binomial expansions of $(1 + \varepsilon_2/\theta_2)^{-1/2}$ and $(1 + \varepsilon_3/\theta_3)^{-1/2}$ as convergent series of ε_2 and ε_3 for large sample size, we get (Gupta/Singh/Lal [2])

$$\begin{aligned} E(r) \doteq \rho \Bigg(1 - \frac{E(\varepsilon_1\varepsilon_2)}{2\theta_1\theta_2} - \frac{E(\varepsilon_1\varepsilon_3)}{2\theta_1\theta_3} + \frac{E(\varepsilon_2\varepsilon_3)}{4\theta_2\theta_3} + \frac{3}{8}\frac{E(\varepsilon_2^2)}{\theta_2^2} \\ + \frac{3}{8}\frac{E(\varepsilon_3^2)}{\theta_3^2} \Bigg) \qquad (16) \end{aligned}$$

where the terms of order $0(n^{-2})$ in (16) have been neglected. Substituting the values of variances and covariances of ε's from the lemmas 2-3 and corollaries 1 — 3, we get bias $B(r) = E(r) - \rho$ as reported in (15). For upper bound of $B(r)$ one can refer to Gupta, Singh and Lal [2].

COROLLARY 7 : For bivariate normal distribution of random variables (x, y), the expected value of r will be approximately ρ.

Proof: For bivariate normal distribution

$$\mu_{20} = \sigma_x^2,\ \mu_{11} = \rho\sigma_x\sigma_y,\ \mu_{02} = \sigma_y^2,$$
$$\mu_{30} = \mu_{03} = \mu_{12} = \mu_{21} = 0,\ \mu_{04} = 3\sigma_y^4,$$
$$\mu_{40} = 3\sigma_x^4,\ \mu_{31} = 3\rho\sigma_x^3\sigma_y,\ \mu_{13} = 3\rho\sigma_x\sigma_y^3$$

and $$\mu_{22} = (1 + 2\rho^2)\,\sigma_x^2\sigma_y^2 \tag{17}$$

Using these relations in the expressions of variances and covariances of ε's in lemmas 2-3 and corollaries 1-3, we get

$$E\frac{(\varepsilon_2^2)}{\theta_2^2} = \frac{\alpha}{\theta_2^2}\{S_{x^2}^2 - 4\overline{X}S_x x^2 + 4\overline{X}^2 S_x^2\} + \frac{2\beta(N-2)}{\theta_2^2(N-3)}S_x^4$$
$$= \frac{\alpha}{\theta_2^2}\left\{\frac{1}{N-1}(2N\sigma_x^4 + 4N\overline{X}^2\sigma_x^2) - \frac{4\overline{X}}{N-1^2}(2N\overline{X}\sigma_x^2) + \frac{4\overline{X}^2N}{N-1}\sigma_x^2\right\} + \frac{2\beta(N-2)}{\theta_2^2(N-3)}\frac{N^2}{(N-1)^2}\sigma_x^4$$
$$= \frac{2N}{N-1}\left(\alpha + \frac{N(N-2)\beta}{(N-1)(N-3}\right)\frac{\sigma_x^4}{\theta_2^2}$$
$$= \frac{2(N-n)(N-1)}{nN(n-1)(N-3)}\left[\frac{nN-N-n-1}{N-2} + \frac{N-n-1}{N-1}\right]$$
$$= 2\left(\frac{1}{n} - \frac{1}{N}\right)\left(\frac{1-1/N}{1-3/N}\right)\frac{1}{(n-1)}\left[\frac{(n-1)-(n+1)/N}{1-2/N} + \frac{1-n+1/N}{1-1/N}\right] \to \frac{2(n-1+1)}{n(n-1)} = \frac{2}{n-1} \text{ as } N\to\infty \tag{18}$$

Similarly $$E(\varepsilon_3^2/\theta_3^2) = \frac{E(\varepsilon_1\varepsilon_2)}{\theta_1\theta_2} = \frac{E(\varepsilon_1\varepsilon_3)}{\theta_1\theta_3} = \frac{2}{n-1} \tag{19}$$

and $$\frac{E(\varepsilon_2\varepsilon_3)}{\theta_2\theta_3} = \frac{2\rho^2}{n-1} \tag{20}$$

From (16), (18), (19) and (20), we get

$$E(r) \doteq \rho\left[1 - \frac{1}{(n-1)} - \frac{1}{(n-1)} + \frac{\rho^2}{2(n-1)} + \frac{3}{8}\left(\frac{2}{n-1} + \frac{2}{n-1}\right)\right]$$
$$\doteq \rho\left[1 - \frac{1-\rho^2}{2(n-1)}\right] \simeq \rho \tag{21}$$

4. The Variance of r

In case of biased estimators, the measure of variation is mean square error. But in case of first order approximation of r, both $V(r)$ and

MSE (r) will be the same. So using the first order approximation of r, variance of r for SRSWOR, is obtained as follows :

THEOREM 2 : *To the first order of approximation*

$$
\begin{aligned}
V(r) \doteq\; & \rho^2\alpha \{S_z^2/\theta_1^2 + S_{x^2}^2 / 4\theta_2^2 + S_{y^2}^2 / 4\theta_3^2 - S_{zx^2}/ \theta_1 \theta_2 \\
& - S_{zy^2} /\theta_1\theta_3 + S_{xy^2}^2 /2\theta_2\theta_3 - (1/\theta_2\theta_3) (\bar{X}S_{xy^2} + \bar{Y}S_{yx^2}) \\
& + (3\bar{X}/\theta_2 - 2\bar{Y}/\theta_1) S_{zx} /\theta_1 + (3\bar{Y}/\theta_3 - 2\bar{X}/\theta_1) S_{zy} /\theta_1 \\
& + (\bar{X}^2/\theta_1^2 + \bar{Y}^2/\theta_3^2 - 3\bar{X}\bar{Y}/\theta_1\theta_3) S_y^2 + (Y^2/\theta_1^2 \\
& + \bar{X}^2/\theta_2^2 - 3\bar{X}\bar{Y}/\theta_1\theta_2) S_x^2 + (2\bar{X}\,\bar{Y}/\theta_1^2 - \bar{X}^2/\theta_1\theta_2 \\
& - \bar{Y}^2/\theta_1\theta_3 + 2\bar{X}\bar{Y}/\theta_3\theta_2) S_{xy} + (\bar{Y}/\theta_1 - \bar{X}\theta_2) S_{xx^2} /\theta_2 \\
& + (\bar{X}/\theta_1 - \bar{Y}/\theta_3) S_{yy^2} /\theta_3\} + (\rho^2\beta/(N-3)) \Bigg\{\Bigg(\frac{N-3}{\theta_1^2} \\
& + \frac{N-1}{\theta_2\,\theta_3} \Bigg) S_{xy}^2 + \left(\frac{N-1}{\theta_1^2} - \frac{1}{\theta_2\,\theta_3}\right) S_x^2 S_y^2 \\
& + (N-2) S_x^4/2\theta_2^2 + (N-2) S_y^4/2\theta_3^2 - \frac{2(N-2)}{\theta_1} (S_x^2/\theta_2 \\
& + S_y^2/\theta_3) S_{xy} \Bigg\}
\end{aligned} \tag{22}
$$

Proof: Using the first order approximation (Gupta, Singh and Lal [2]), we get

$$
\begin{aligned}
V(r) = \rho^2 [& E(\varepsilon_1^2)/\theta_1^2 + E(\varepsilon_2^2)/(4\theta_2^2 + E(\varepsilon_3^2)/4\theta_3^2 - E(\varepsilon_1 \varepsilon_2)/\theta_1 \theta_2 \\
& - E(\varepsilon_1 \varepsilon_3)/\theta_1 \theta_3 + E(\varepsilon_2 \varepsilon_3)/2\theta_2 \theta_3]
\end{aligned} \tag{23}
$$

Using values of variances and covariances of ε's from lemmas 1-3 and corollaries 1-3 in (23) and rearranging them, we get (22).

COROLLARY 8 : For random variables (x, y) following a bivariate normal distribution, the variance, $V(r)$ of r is approximately $(1-\rho^2)^2/(n-1)$

From lemma 1, we have

$$
\begin{aligned}
E(\varepsilon_1^2)/\theta_1^2 = & (\alpha/\theta_1^2) (S_z^2 - 2\bar{X}S_{zy} - 2\bar{Y}S_{zx} + \bar{X}^2 S_y^2 + \bar{Y}^2 S_x^2 \\
& + 2\bar{X}\bar{Y}S_{xy}) + (\beta/\theta_1^2)\left(S_{xy}^2 + \frac{N-1}{N-3} S_x^2 S_y^2 \right)
\end{aligned} \tag{24}
$$

Using relations in (17) in the equation (24), we get

$$
\begin{aligned}
E(\varepsilon_1^2)/\theta_1^2 = (1/\theta_1^2)\Bigg[& \frac{\alpha N}{N-1} (1 + \rho^2) \sigma_x^2 \sigma_y^2 + \frac{\beta N^2 \sigma_x^2 \sigma_y^2}{N-1} \\
& \left(\frac{\rho^2}{N-1} + 1/(N-3) \right)\Bigg]
\end{aligned}
$$

$$= \frac{(N-n)(N-1)}{nN(n-1)(N-2)} (1/\rho^2) \left[\left(\frac{nN-N-n-1}{N-3} + \frac{N-n-1}{N-1}\right)\rho^2 + \frac{nN-2n-2}{N-3}\right]$$

$$= (1/n - 1/N) \frac{1-1/N}{1-2/N} \frac{1}{(n-1)\rho^2} \left[\left(\frac{n-1-(n+1)/N}{1-3/N} + 1 - \frac{n}{N-1}\right)\rho^2 + \frac{n-2n+2/N}{1-3/N}\right] \to \frac{1+\rho^2}{(n-1)\rho^2}$$

$$\text{as } N \to \infty \quad (25)$$

Substituting values of $E(\varepsilon_i \varepsilon_j)/\theta_i \theta_j$ $(i, j = 1, 2, 3)$ from (18), (19), (20) and (25) in the expression of $V(r)$ given in (23), we get

$$V(r) \doteq \rho^2 \left(\frac{1+\rho^2}{(n-1)\rho^2} + \frac{1}{2(n-1)} + \frac{1}{2(n-1)} - \frac{2}{n-1} - \frac{2}{n-1} + \frac{\rho^2}{n-1}\right) = \frac{1}{n-1}(1-\rho^2)^2 \quad (26)$$

Values of $E(\varepsilon_i \varepsilon_j)/\theta_i \theta_j$ $(i, j = 1, 2, 3)$ computed by Gupta, Singh and Lal [2], for a bivariate normal distribution, are wrong though due to contrast nature of expression of $B(r)$ computed fron (16) and that of $V(r)$ in (23), they obtained the same values as reported in this paper.

Similarly, we get expressions for the bias and variance of r for SRSWR by simply substituting values of variances and covariances of ε's from lemmas 4·6 and corollaries 4·6 in (16) and (23), respectively and also replacing θ_i $(i = 1, 2, 3)$ with $(N - 1/N)\,\theta_i$. It can be easily observed that the multiplier $n(n-1)(N-1)^2/n^2N^6$ reported by Gupta/Singh/Lal [2] in the expressions of bias and variance of r should be replaced by $(n-1)/nN^2(N-1)^2$.

5. Estimation of the Variance of r

To avoid mathematical complexity, a practically useful biased estimator of $V(r)$ is provided by replacing θ_i $(i = 1, 2, 3)$, $\bar{X}$, $\bar{Y}$, S_a^2 add S_{ab} $(a, b = z, x, y, x^2, y^2)$ with $\hat{\theta}_i$, $\bar{x}$, $\bar{y}$, s_a^2 and s_{ab}, respectively in the expression of $V(r)$ given in (22), The estimator so obtained is given below ;

THEOREM 3 : *For SRSWOR, an estimator of $V(r)$ is given by*

$$\hat{V}(r) = r^2 \alpha \{s_z^2/(\hat{\theta}_1)^2 + s_{x^2}^2/4(\hat{\theta}^2)^2 + s_{y^2}^2/4(\hat{\theta})_3 - s_{zx^2}/\hat{\theta}_1\hat{\theta}_2 - s_{zy^2}/\hat{\theta}_1\hat{\theta}_3 + s_{x^2y^2}/2\hat{\theta}_2\hat{\theta}_3 - \frac{1}{\hat{\theta}_2\hat{\theta}_3}(\bar{x}\, s_{xy^2} + \bar{y}\, s_{yx^2})$$

$$+ (3\bar{x}/\hat{\theta}_2 - 2\bar{y}/\hat{\theta}_1)\, s_{zx}/\hat{\theta}_1 + (3\bar{y}/\hat{\theta}_3 - 2\bar{x}/\hat{\theta}_1)\, s_{zy}/\hat{\theta}_1$$
$$+ (\bar{x}^2/(\hat{\theta}_1)^2 + \bar{y}^2/(\hat{\theta}_3)^2 - 3\bar{x}\bar{y}/\hat{\theta}_1\hat{\theta}_3)\, s_y^2$$
$$+ (\bar{y}^2/(\hat{\theta}_1)^2 + \bar{x}^2/(\hat{\theta}_2)^2 - 3\bar{x}\bar{y}/\hat{\theta}_1\hat{\theta}_2)\, s_x^2$$
$$+ (2\bar{x}\bar{y}/(\hat{\theta}_1)^2 - \bar{x}^2/\hat{\theta}_1\hat{\theta}_2 - \bar{y}^2/\hat{\theta}_1\hat{\theta}_3 + 2\bar{x}\bar{y}/\hat{\theta}_2\hat{\theta}_3)\, s_{xy}$$
$$+ (\bar{y}/\hat{\theta}_1 - \bar{x}/\hat{\theta}_2)\, s^2_{xx}/\hat{\theta}_2 + \bar{x}/\hat{\theta}_1 - \bar{y}/\hat{\theta}_3)\, s_{yy}\, 2/\hat{\theta}_3\}$$
$$+ \frac{r^2\beta}{(N-3)}\left\{\left(\frac{N-3}{(\hat{\theta}_1)^2} + \frac{N-1}{\hat{\theta}_2\hat{\theta}_3}\right) s^2_{xy} + \left(\frac{N-1}{(\hat{\theta}_1)^2} - \frac{1}{\hat{\theta}_2\hat{\theta}_3}\right)\right.$$
$$s_x^2 s_y^2 + (N-2)\, s_x^4/2\,(\hat{\theta}_2)^2 + (N-2)\, s_y^4/2(\hat{\theta}_3)^2$$
$$\left. - \frac{2\,(N-2)}{\hat{\theta}_1} \cdot (s_x^2/\hat{\theta}_2 + s_y^2/\hat{\theta}_3)\, s_{xy}\right\} \qquad (27)$$

Similarly, an estimator of $V(r)$ can be obtained for SRSWR by replacing, $\bar{X}$, $\bar{Y}$, θ_i $(i = 1, 2, 3)$, σ_a^2 and σ_{ab} $(a, b = z, x, y, x^2, y^2)$ with their unbiased estimators, $\bar{x}$, $\bar{y}$, $N\hat{\theta}_i/N-1$, s_a^2 and s_{ab}, respectively in the expression of $V(r)$ obtained for SRSWR.

ACKNOWLEDGEMENT

The author is thankful to the referees for their valuable suggestions in the original manuscript of the paper.

REFERENCES

[1] Elderton, W. P. (1953) : *Frequency Curves and Correlation* (4th edition). Cambridge University Press, London.

[2] Gupta, J. P., Singh, R. and Lal, B. (1978) : On the estimation of the finite population correlation coefficient—I. *Sankhya* **40** : 38-59.